OPTIMAL FORMATION RECONFIGURATION OF SATELLITES WITH ATTITUDE CONSTRAINTS USING THRUSTERS

Yasuhiro Yoshimura\textsuperscript{(1)} and Shinji Hokamoto\textsuperscript{(2)}

\textsuperscript{(1)(2)}Kyushu University, 744 Motooka Nishi-ku Fukuoka Japan, +81 92 802 3000, y-yas@aero.kyushu-u.ac.jp

\textbf{Abstract:} An optimal reconfiguration control for satellites formation flying with attitude constraints by the use of a small number of thrusters is addressed. These thrusters are fixed to the satellite body and generate unilateral forces. This study assumes that the satellite’s attitude angle during the reconfiguration maneuver is restricted within a certain bound from a specified direction in an inertial frame. Such constraint arises from some requirements: electric power generation with fixed solar array panels, observation or communication with ground-based stations, and so on. Furthermore, the satellite is assumed to have a few number thrusters; for in-plane formation maneuver the number of thrusters is set to two. This latter assumption reduces the number of actuators required for formation flying, and thus this study is useful when several thrusters of a satellite fail due to malfunctions. The purpose of this paper is to derive an optimal controller to reconfigure the formation between two satellites by using a few number of thrusters under the satellite’s attitude constraints. To this end, firstly a control method to track reference inputs is derived, and then the reference input design and the condition to satisfy the attitude constraint are discussed. Numerical simulation results show that the designed reconfiguration method is effective for keeping the satellite attitude angle within a specified bound.

\textbf{Keywords:} Optimal Reconfiguration, Attitude Constraint, Thrusters.

1. Introduction

Optimal reconfiguration or rendezvous problems of satellites formation have been discussed in many studies to minimize fuel or energy consumption for a relative orbit control of a follower satellite. The relative motion of a follower in the proximity of a leader is expressed with Clohessy-Wiltshire equation or Tchauner-Hempel equation, respectively, for a circular and an elliptic orbit of the leader. These equations have periodic solutions and the solutions are useful to design a satellite formation or a rendezvous trajectory. From the viewpoint of practical missions, maneuvers with less fuel consumption are required. In terms of energy optimality, Carter and Pardis\textsuperscript{[1]} derived an optimal feedback controller for a rendezvous problem in a circular orbit with bounded magnitudes of thrusts. Palmer\textsuperscript{[2]} shows an analytical solution to relocate a follower satellite to a desired relative orbit using the Fourier series. The method is extended to discuss the reachability of a reconfiguration problem with bounded inputs\textsuperscript{[3]} as well as to derive an optimal reconfiguration method for a formation flying in an elliptic orbit\textsuperscript{[4]}. Xi and Li\textsuperscript{[5]} also show an optimal reconfiguration controller in an elliptic orbit, and both energy and fuel optimality are derived based on a homotopic approach.
In practice, an attitude control of a follower satellite during a reconfiguration maneuver should be considered for some requirements: electric power generation with fixed solar array panels, observation or communication with ground stations, and so on. Moreover, these requirements are equivalent to the restriction of thrust directions when some thrusters have failed or a satellite equips a few number of thrusters. In most of studies for formation flying enough number of thrusters is assumed to be available, and few papers discuss the attitude control and the constraint of input directions which arise from a restricted number of thrusters. Mitani and Yamakawa[6] show a rendezvous procedure under constraints of thruster directions with respect to a leader satellite. The paper utilizes modal analysis and derives an optimal controller based on a satisficing method to keep the thrust direction within an allowable area. The method is further extended to an optimal controller in terms of $L_1$ and $L_2$ norm of thruster accelerations using a smoothing method[7]. Guelman et al.[8] deal with a formation control under a single input constraint in a circular orbit assuming the satellite’s attitude can be controlled arbitrarily.

The purpose of this paper is to show an optimal reconfiguration method by using thruster forces under a satellite’s attitude constraint with respect to an inertial frame. Furthermore, the satellite is assumed to have a few number of thrusters; for in-plane formation maneuver the number of thrusters is set to two. This assumption reduces the number of actuators required for formation flying, and thus this study is useful when several thrusters of a satellite fail due to malfunctions. For a satellite equips a small number of thrusters, the satellite attitude must be controlled to generate a thruster force in a desired direction. In this paper, firstly it is shown that thruster inputs can control the satellite to track reference accelerations based on the Lyapunov stability. The tracking controller reduces the reconfiguration problem under attitude constraints to the one under thrust directional constraints. Then, the reference input design and conditions to satisfy the constraint on the thrust direction is discussed using the controller shown in [3].

This paper is organized as follows. Section 2 denotes the relative equations of a follower satellite in a circular orbit and derives the analytic solution. Modal equations are also shown to simplify the interpretation of reconfiguration problem. The next section firstly describes that the satellite can track a reference trajectory using thruster forces considering the attitude change. The optimal reference input is designed by following the control method derived in [3], and then, the condition to satisfy the attitude constraint is obtained for the reference input. Finally, numerical simulation results are shown in section 4.

2. Equations of motion

2.1. Hill’s equation

The equations of motion for formation flying in a circular orbit have been studied based on the linearized equation, called Hill’s equation. The Hill’s equation describes the relative motion of a follower satellite orbiting near a leader. In the leader-fixed
coordinate, $x$-axis lies in the radial direction from the Earth, $z$-axis points to the orbital momentum vector of the leader, and $y$-axis completes a right-handed coordinate. The equation of motion of the follower satellite is written as

$$\dot{R} = -\frac{\mu}{(R_0 + x)^{3/2}} (R_0 + x) + u$$

(1)

where $R, R_0$ and $\mu$ are an orbital radius of a follower, an orbital radius of a leader, and a gravitational constant. Also, $x = \begin{bmatrix} x & y & z \end{bmatrix}^T$ and $u = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T$ denote a relative position of a follower and acceleration inputs, respectively. Assuming the orbital radius of the leader satellite is much larger than the distance between the leader and a follower, we obtain linearized equations as

**In-plane motion**

$$\begin{bmatrix} \Omega_x \\ \dot{x} \\ \Omega_y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & \Omega & 0 & 0 \\ 3\Omega & 0 & 0 & 2\Omega \\ 0 & 0 & 0 & \Omega \\ 0 & -2\Omega & 0 & 0 \end{bmatrix} \begin{bmatrix} \Omega_x \\ \dot{x} \\ \Omega_y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

(2)

$$\Rightarrow \dot{x} = Ax + Bu_{xy}$$

**Out-of-plane motion**

$$\frac{d}{dt} \begin{bmatrix} \Omega_z \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} \begin{bmatrix} \Omega_z \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 \\ u_z \end{bmatrix}$$

(3)

where $\Omega$ is an orbital rate of the leader satellite. Since the out-of-plane motion is decoupled with the in-plane motion and is simple harmonic, this paper considers a reconfiguration problem in $x$-$y$ plane.

**2.2. Analytic Solution**

The analytic solution of Hill’s equation with no external forces is obtained as follows.

$$x_h(t) = \begin{bmatrix} 4 - 3c & s & 0 & 2(1-c) \\ 3s & c & 0 & 2s \\ 6(s - \Omega t) & -2(1-c) & 1 & 4s - 3t \\ -6(1-c) & -2s & 0 & -3 + 4c \end{bmatrix} x_0$$

(4)

$$\Rightarrow x_h(t) = \Phi(t)x_0$$
where \( c := \cos \Omega t \), \( s := \sin \Omega t \) and the subscript “0” denotes the initial state. Furthermore, Eq. 4 is simplified as

\[
\begin{align*}
x(t) &= a \cos(\Omega t + \phi) + 2b \\
y(t) &= -2a \sin(\Omega t + \phi) - 3bt + d \\
x'(t) &= -\Omega a \sin(\Omega t + \phi) \\
y'(t) &= -2\Omega a \cos(\Omega t + \phi) - 3b
\end{align*}
\]

where

\[
\begin{align*}
a &:= \sqrt{(3x_0 + 2\dot{y}_0 / \Omega)^2 + (\ddot{x}_0 / \Omega)^2} \\
b &:= 2\Omega x_0 + \dot{y}_0 \\
d &:= y_0 - 2\dot{x}_0 / \Omega \\
\phi &:= \arctan \left( \frac{\ddot{x}_0}{\Omega \left(3x_0 + 2\dot{y}_0 / \Omega\right)} \right)
\end{align*}
\]

Since the follower position forms

\[
\left( \frac{x - 2b}{a} \right)^2 + \left( \frac{y + 3bt - d}{2a} \right)^2 = 1
\]

the relative motion of the follower becomes an elliptic orbit when \( b = 0 \) and a leader-centered ellipse when \( b = d = 0 \). Thus, the parameters, \( a \), \( b \), \( d \), and \( \phi \) denote the size of relative orbit, a drift velocity, the center distance of the ellipse from the leader satellite, and an initial phase, respectively.

### 2.3 Modal Equation

A variable transformation based on modal analysis can simplify the relative motion of a follower satellite[3][8]. However, the system matrix, \( A \), for the in-plane motion of Hill’s equation is defective and thus only three eigenvectors and eigenvalues are obtained, although the order of \( A \) is four. The eigenvectors and eigenvalues are calculated as

\[
\begin{bmatrix} e_1 & e_3 & e_4 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & -1/2 \\ 1 & i/2 & -i/2 \\ 0 & i & -i \\ 0 & 1 & 1 \end{bmatrix}
\]

\[\lambda_1 = 0, \ \lambda_3 = -i\Omega, \ \lambda_4 = i\Omega\]
where \( i \) is an imaginary number. The following operation forms two real eigenvectors.

\[
e'_{3} = \Omega(e_{3} - e_{4}) / i = \begin{bmatrix} 0 & 1 & 2 & 0 \end{bmatrix}^T
\]

\[
e'_{4} = \Omega(e_{3} + e_{4}) = \begin{bmatrix} -1 & 0 & 0 & 2 \end{bmatrix}^T
\]

where the superscript “\( T \)” denotes a transpose of the vector or matrix. A generalized eigenvector \( e_{2} \) is obtained as follows.

\[
(A - \lambda_{i} I) e_{2} = e_{1} \\
\Rightarrow e_{2} = \begin{bmatrix} -2 / \Omega & \alpha & 0 & 3 / \Omega \end{bmatrix}^T
\]

where \( \alpha \) is an arbitrary value and henceforth the case when \( \alpha = 0 \) is considered for simplicity. The modal variables are defined using a transformation matrix consists of the eigenvectors.

\[
\xi = \begin{bmatrix} \xi_{1} & \xi_{2} & \xi_{3} & \xi_{4} \end{bmatrix}^T \equiv P \mathbf{x}
\]

where

\[
P := E^{-1} = \begin{bmatrix} -e'_{4} & e'_{3} & -e_{1} & -e_{2} \end{bmatrix}^{-1}
\]

and equivalently,

\[
\begin{align*}
\xi_{1} &= -3\Omega x - 2\dot{y} \\
\xi_{2} &= \dot{x} \\
\xi_{3} &= 2\dot{x} - \Omega y \\
\xi_{4} &= 2\Omega^2 x + \Omega \dot{y}
\end{align*}
\]

The modal variables \( \xi_{1} \) and \( \xi_{2} \) denote an oscillatory mode, whereas, \( \xi_{3} \) means the distance between the leader and the center of relative elliptic orbit, and \( \xi_{4} \) is the drift velocity, respectively. In fact, the Euclidean norm of \( \Omega \xi_{1} \) and \( \Omega \xi_{2} \) has the same form as the parameter \( a \) shown in Eq. 9; the initial values of \( \xi_{1} \) and \( \xi_{4} \) are equivalent to the parameters \( -\Omega d \) and \( \Omega b \) defined in Eqs. 11 and 10. The differential equations of the modal variables are described as follows:
\[
\dot{\xi} = PAP^{-1}\xi + PBu_{xy}
\]
\[
\Rightarrow \dot{\xi} = \begin{bmatrix}
0 & \Omega & 0 & 0 \\
-\Omega & 0 & 0 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix} \xi + \begin{bmatrix}
1 & 0 \\
0 & 2 \\
-2 & 0 \\
0 & -\Omega
\end{bmatrix} u_{xy}
\]

(25)

The state transition matrix \( \Phi \) is also simplified with the modal variables as follows.

\[
P\Phi P^{-1} = \begin{bmatrix}
\cos \Omega t & \sin \Omega t & 0 & 0 \\
-\sin \Omega t & \cos \Omega t & 0 & 0 \\
0 & 0 & 1 & 3\Omega t \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(26)

Equations 25 and 26 indicate that the oscillatory mode described with \( \xi_1 \) and \( \xi_2 \) are decoupled with the drift motion terms \( \xi_3 \) and \( \xi_4 \).

### 2.4 Rotational Equation and Thruster Configuration

The rotational equation of a follower in \( x-y \) plane is expressed with a single spin motion as

\[
\psi = \omega_z - \Omega
\]

(27)

\[
J_z \dot{\omega}_z = T_z
\]

(28)

where \( \psi, \omega_z, T_z \) and \( J_z \) are an attitude angle in the leader-fixed frame, an angular rate, an external torque, and the moment of inertia around \( z \)-axis, respectively.

In the present paper, we assume that the follower satellite equips two thrusters to control the position and attitude and, without loss of generality, the \( x \)-axis in body-fixed frame corresponds with the thrust direction as shown in Fig. 1. The thrusters can generate variable magnitudes of thrust forces whose directions are restricted only in positive direction due to thruster mechanisms. Thus, the external forces shown in Eq. 2 are written in the following relation.

\[
u_{xy} = \frac{1}{m} R(\psi)F_b
\]

(29)

where \( m \) and \( F_b = \begin{bmatrix} f_1 + f_2 & 0 \end{bmatrix}^T (f_1, f_2 \geq 0) \) are the satellite mass and the thruster forces in the body-fixed frame, respectively, and \( R(\cdot) \) denotes a rotational matrix, that is,
\[ R(\psi) = \begin{bmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{bmatrix} \] 

(30)

The control torques, on the other hand, is written as

\[ T_z = f_1 \beta_1 + f_2 \beta_2 \] 

(31)

where \( \beta_1 \) and \( \beta_2 \) are the moment arms of thrusters which take a negative value when the thruster generates a clockwise directional control torque. Consequently, the following relation is used to distribute the required control acceleration and torque into two thrusters.

\[
\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{bmatrix}^{-1} \begin{bmatrix} m \sqrt{u_x^2 + u_y^2} \\ T_z \end{bmatrix}
\] 

(32)

Although the inverse matrix in the right-hand side of Eq. 32 becomes singular when \( \beta_1 = \beta_2 \), this singularity can be ignored because such condition occurs when both thrusters generate control torques in the same direction, i.e. the system becomes uncontrollable.

![Figure 1. Thruster configuration](image_url)

3. Control Method

This section deals with an optimal reconfiguration method under attitude constraints. As shown in Fig. 2, we assume that the attitude constraint of the follower is less than a specified angle \( \psi_{\text{bound}} \) from \( X \)-axis in the inertial coordinates and the desired attitude angle, e.g. a normal vector of a solar array panel or an antenna, is expressed with \( \psi_{\text{offset}} \) in the body-fixed frame. Then the attitude constraint is written as

\[ |\theta(t) + \psi(t) + \psi_{\text{offset}}| \leq \psi_{\text{bound}} \] 

(33)
where $\theta$ denotes a true anomaly of leader satellite. In the reconfiguration maneuver, the size of relative orbit of follower is controlled to a desired one, and thus the reconfiguration problem in terms of the modal variables is described as 

$$\begin{pmatrix} \xi_1, \xi_2, \xi_3, \xi_4 \end{pmatrix} \rightarrow \begin{pmatrix} \xi_{1d}, \xi_{2d}, \xi_{3d}, \xi_{4d} \end{pmatrix}. $$

3.1 Reference Input Tracking

A control method to track reference accelerations is firstly derived. The satellite’s attitude must be controlled so that the thrust direction is oriented along the reference acceleration vector. To this end, the following Lyapunov function candidate is considered.

$$L = \frac{1}{2}(\psi - \psi_d)^2 + \frac{1}{2}(\dot{\psi} - \dot{\psi}_d)^2$$  \hspace{1cm} (34)

where $\psi_d := \arctan\left(\frac{u_{ad}(t)}{u_{ad}(t)}\right)$ representing the angle of the reference acceleration vector. The time derivative of Eq. 34 is calculated as

$$\dot{L} = (\dot{\psi} - \dot{\psi}_d)(\ddot{\psi} - \ddot{\psi}_d + \psi - \psi_d)$$

$$= (\dot{\psi} - \dot{\psi}_d)(T / J_z - \dot{\psi}_d + \psi - \psi_d)$$  \hspace{1cm} (35)

Thus, the following controller is proposed.

$$T / J_z = \dot{\psi}_d - \dot{\psi} + \psi_d - K_\psi (\psi - \dot{\psi}_d)$$  \hspace{1cm} (36)

where $K_\psi$ is a positive constant gain. Substituting the control input into Eq. 35, we obtain the time derivative of Lyapunov function as

$$\dot{L} = -K_\psi (\psi - \dot{\psi}_d)^2 \leq 0$$  \hspace{1cm} (37)
Therefore, the control torque shown in Eq. 36 drives the follower attitude to track the reference acceleration. This indicates that the reconfiguration problem under the attitude constraint is equivalent to the one under the directional constraint of the reference acceleration. Also, note that the reference inputs, $u_{xd}$ and $u_{yd}$, must be at least two times differentiable due to the term $\ddot{\psi}_d$. If the reference inputs $u_{xd}$ or $u_{yd}$, for instance, include a feedback term of a velocity $\dot{x}$, the term $\ddot{\psi}_d$ requires the time derivative of an acceleration $\dddot{x}$ for the feedback. Such term is difficult to estimate and thus the reference tracking method disables an application of a full-state feedback, e.g. LQR controller.

3.2 Optimal Reconfiguration Controller

The optimal controller for the reference inputs under the directional constraint is discussed based on the method shown in [2]. The advantages of using the controller in [2] are: 1) the optimal input is infinitely differentiable because of the function of time; 2) the condition to satisfy the attitude constraint is obtained from an analogy with the analytic solution of Hill’s equation. The control method in [2] is here briefly followed.

Regarding accelerations as control inputs, the optimal controller is designed to minimize the energy consumption. The cost function is defined for a maneuver with a fixed time $t_f$ as

$$J = \int_0^{t_f} \left( u_x^2 + u_y^2 \right) dt$$

(38)

The control inputs are expressed with the Fourier series as:

$$u_x = \frac{a_x0}{2} + \sum_{n=1}^{\infty} \left( a_xn \cos(n\Omega_f t) + b_xn \sin(n\Omega_f t) \right)$$

(39)

$$u_y = \frac{a_y0}{2} + \sum_{n=1}^{\infty} \left( a_yn \cos(n\Omega_f t) + b_yn \sin(n\Omega_f t) \right)$$

(40)

where $\Omega_f = 2\pi / t_f$, and $a_{j0}$, $a_{jn}$ and $b_{jn}$ ($j = x, y$) are the Fourier coefficients. The cost function is rewritten in terms of the Fourier coefficients using the Parseval’s theorem as follows.

$$J = \frac{t_f}{2} \left( \frac{a_x0^2}{2} + \sum_{n=1}^{\infty} (a_xn^2 + b_xn^2) \right) + \frac{t_f}{2} \left( \frac{a_y0^2}{2} + \sum_{n=1}^{\infty} (a_yn^2 + b_yn^2) \right)$$

(41)

Thus, the optimal control problem is equivalent to finding the Fourier coefficients which minimize the cost function, and the Fourier coefficients are related to the boundary condition and the Lagrange multipliers.
The analytic solution with control inputs are described as

\[ x(t_f) = x_h(t_f) + x_p(t_f) = \Phi x_0 + x_p(t_f) \]  \hspace{1cm} (42)

where \( x_p(t_f) \) is a particular solution. Furthermore, the term of particular solution is rewritten as a matrix form:

\[
x_p(t_f) = \begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -3\Omega t_f & 0 & 4 & 3 \\ -3 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix}
\]

\[ \Rightarrow x_p(t_f) = B_p I \]  \hspace{1cm} (43)

where

\[ I_2 = \int_0^{t_f} u_y(\tau) d\tau \]

\[ I_3 = \int_0^{t_f} u_y(\tau) \cos[\Omega(t_f - \tau)] d\tau - \frac{1}{2} \int_0^{t_f} u_x(\tau) \sin[\Omega(t_f - \tau)] d\tau \]

\[ I_4 = \int_0^{t_f} u_y(\tau) \sin[\Omega(t_f - \tau)] d\tau + \frac{1}{2} \int_0^{t_f} u_x(\tau) \cos[\Omega(t_f - \tau)] d\tau \]

\[ I_5 = \Omega \int_0^{t_f} u_y(\tau) \tau d\tau - \frac{2}{3} \int_0^{t_f} u_x(\tau) d\tau \]  \hspace{1cm} (47)

The integral terms describe the constraints between a desired state and a homogeneous solution as

\[
\begin{bmatrix} I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = B_p^{-1} (x(t_f) - x_h(t_f)) = 
\begin{bmatrix} 2 & 0 & 0 & 1 \\ 3/2 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & 0 \\ 2\beta & -2/3 & 1/3 & \beta \end{bmatrix} 
\begin{bmatrix} \Omega(x(t_f) - x_h(t_f)) \\ \dot{x}(t_f) - \dot{x}_h(t_f) \\ \Omega(y(t_f) - y_h(t_f)) \\ \dot{y}(t_f) - \dot{y}_h(t_f) \end{bmatrix}
\]

\[ \text{Eq. 48} \]

The constraints of boundary states are transformed with the modal variables shown in Eq. 19 and rewritten as

\[ I_{\xi} = B_p^{-1} P^{-1} (\xi - \xi_h) \]  \hspace{1cm} (49)

The permutation of the components of \( I_{\xi} \) yields \( I'_{\xi} = [ I_{\xi 3} \quad I_{\xi 4} \quad I_{\xi 5} \quad I_{\xi 2} ]^T \) and this constraint is further transformed as
where $K = \begin{bmatrix} K_3 \sin(\beta/2) & K_4 \sin(\beta/2) & K_5 & K_2 \end{bmatrix}^T$ and $\beta = \Omega t_f$. The transformed constraint $K$ provides the simple relationship with the Lagrange multipliers as shown in [2]. The optimal inputs for a fixed time reconfiguration are described as follows.

$$u_x(t) = \frac{2}{3} T_1 + \Lambda \sin(\Omega t - \Theta)$$

$$u_y(t) = T_0 - T_1 \Omega t + \Lambda \cos(\Omega t - \Theta)$$

where $T_0$, $T_1$, $\Lambda$ and $\Theta$ are constants described with the Lagrange multipliers.

### 3.3 Condition on Attitude Constraints

The conditions to satisfy the attitude constraint are obtained from an analogy between the homogeneous solution of Hill’s equation and the optimal controller. The optimal controller shown in Eqs. 51 and 52 have similar forms to the simplified Hill’s solutions in Eqs. 5 and 6, respectively. This analogy means that the time history of the optimal input is similarly described as a free relative motion of the follower. Thus, the input trajectory is predicted to be an ellipse when $T_1 = 0$ and furthermore an origin-centered ellipse when $T_0 = T_1 = 0$. Since the leader satellite is orbiting in a circular orbit, the desired attitude angle represented in the leader-fixed frame monotonically decreases at the rate of $\Omega$. Therefore, the analogy is useful to design the input trajectory satisfying the attitude constraint.

The relation between an initial state and a target state provides the conditions to make an input trajectory an ellipse, i.e. $T_1 = 0$. Since the boundary states of $\xi_3$ and $\xi_4$ are $\xi_{3,0} = \xi_{3b}(t_f)$ and $\xi_{4,0} = \xi_{4b}(t_f)$, the integral constraints are simplified to $K_2 = K_5 = 0$ from Eqs. 49 and 50. Also, the cost function is further minimized when $\Theta = \beta / 2$ [3]. These boundary states indicate that the condition, $T_1 = 0$, is satisfied when $K_4 = 0$. Thus, using Eqs. 49 and 50, we obtain the boundary constraint as follows.
\[(\xi_i(t_f) - \xi_i(t_h))\tan\frac{\beta}{2} + (\xi_2(t_f) - \xi_2(t_h)) = 0\]  

(53)

The modal variables are here rewritten with polar coordinates as

\[
\xi_i(t_f) = a_f \cos \gamma_f \\
\xi_2(t_f) = a_f \sin \gamma_f
\]

(54, 55)

where the notation \((t_f)\) is dropped off for the sake of simplicity and instead the subscript "f" denotes the states when \(t = t_f\). Similarly, using the state transition matrix of the modal variables in Eq. 26, we obtain

\[
\xi_i(t_h) = a_0 \cos \gamma_0 \cos \beta + a_0 \sin \gamma_0 \sin \beta \\
\xi_2(t_h) = -a_0 \cos \gamma_0 \sin \beta + a_0 \sin \gamma_0 \cos \beta
\]

(56, 57)

Substitution of Eqs. 54-57 into Eq. 53 yields

\[
a_f \left(\tan\frac{\beta}{2} \cos \gamma_f + \sin \gamma_f\right) - a_0 \left(\sin(\gamma_0 - \beta) + \tan\frac{\beta}{2} \cos(\gamma_0 - \beta)\right) = 0
\]

(58)

Since the parameters \(a_0\) and \(a_f\) are nonzero and determined by the initial and the final states, the following conditions are obtained:

\[
\tan\frac{\beta}{2} \cos \gamma_f + \sin \gamma_f = 0
\]

(59)

\[
\sin(\gamma_0 - \beta) + \tan\frac{\beta}{2} \cos(\gamma_0 - \beta) = 0
\]

(60)

These equation hold when the following boundary states are satisfied for the reconfiguration maneuver with a fixed time \(\beta = \Omega t_f\).

\[
\frac{y_0}{2x_0} = \tan\frac{\beta}{2}
\]

(61)

\[
\frac{y_f}{2x_f} = \tan\left(-\frac{\beta}{2}\right)
\]

(62)

Therefore, the initial and the final states satisfying Eqs. 61 and 62 make \(T_i = 0\), and the optimal input trajectory becomes an ellipse.

4. Numerical Simulation

Numerical simulation demonstrates the effectiveness of the reconfiguration maneuver under the attitude constraints. The satellite mass, the moment of inertia, and the
moment arm of thrusters are set as $m = 200.0 \text{[kg]}$, $J_z = 60 \text{[kgm}^2\text{]}$, and $(\beta_1, \beta_2) = (0.5, -0.5)\text{[m]}$. The leader satellite is assumed to be orbiting in a circular orbit at $6.313 \times 10^{-4} \text{ rad/s}$. Also, the follower is controlled to the target orbit from the initial semimajor axis 4000 m to the target one 2000 m. The parameters for the attitude constraint are considered as $\psi_{\text{offset}} = 0 \text{[deg]}$ and $|\psi_{\text{bound}}| = 45 \text{[deg]}$.

Figure 3 describes the reconfiguration trajectory of the follower satellite and shows the follower is successfully controlled to the target orbit. Figures 4 and 5 show the time history of the attitude angle with respect to the inertial frame and the trajectory of inputs, respectively. The attitude angle of the follower is kept less than the specified bound during the maneuver, and the input trajectory describes an ellipse discussed by the analogy with the analytic solution of Hill’s equation.

![Figure 3. Reconfiguration trajectory of the follower](image)

5. Conclusions

This paper has dealt with an optimal reconfiguration problem in a circular orbit using a few number of thrusters under attitude constraints. Firstly, the attitude tracking method to follow a reference orbit is derived based on the Lyapunov stability. The tracking controller reduces the reconfiguration problem under the attitude constraint to the one under the thrust directional constraint. Thus, using the analogy between the optimal controller with the Fourier series and the analytical solution of Hill’s equation, we show the condition to keep the attitude angle less than a specified bound. Numerical simulation demonstrates the verification of the proposed method.
6. References


